Why the sum of the integers does not equal -1/12

The assertion in the YouTube clip

The assertion is that

$$1 + 2 + 3 + 4 + 5 + 6 + ... = -\frac{1}{12}$$

with reasoning something like:

Let

$$\begin{array}{lccccccc} S_1 & = & 1 & -1 & +1 & -1 & +1 & -1 & ... \\ S_2 & = & 1 & -2 & +3 & -4 & +5 & -6 & ... \\ S & = & 1 & +2 & +3 & +4 & +5 & +6 & ... \end{array}$$

We add S_2 to itself, displacing the terms one to the right as shown below, to obtain

$$\begin{array}{lcccccccc} 2 \times S_2 & = & 1 & -2 & +3 & -4 & +5 & -6 & ... \\ & & & +1 & -2 & +3 & -4 & +5 & ... \\ & = & 1 & -1 & +1 & -1 & +1 & -1 & ... \end{array}$$

$$2 \times S_2 = S_1 \tag{1}$$

Now subtract S_2 from S as follows

$$\begin{array}{lcccccccc} S - S_2 & = & 1 & +2 & +3 & +4 & +5 & +6 & ... \\ & -(& 1 & -2 & +3 & -4 & +5 & -6 & ...) \\ & = & & 4 & & +8 & & +12 & ... = 4S \end{array}$$

$$3S = -S_2 \tag{2}$$

Giving $S = -\frac{S_2}{3} = -\frac{S_1}{6}$

S_1 oscillates between 0 and 1 so on average its value is 1/2, giving

$$S = 1 + 2 + 3 + 4 + 5 + 6... = \frac{-1}{12} \tag{3}$$

This is complete bullshit as explained below. I hope it was created as a hoax, maybe April 1st?. But first, let's assume the reasoning is OK and let's use similar reasoning to see what we can "prove" about $S_2 = 1 - 2 + 3 - 4 + 5 - 6....$ We can arrange the terms in 2 different ways to get:

$$S_2 = (1 - 2) + (3 - 4) + (5 - 6) + ... = -1 - 1 - 1 - ... = -\infty$$

or

$$S_2 = 1 - (2 - 3) - (4 - 5) - ... = 1 + 1 + 1 + 1 + ... = \infty$$

So we've now "proved" $S_1 = \infty$ and $S_1 = -\infty$! Something must have gone wrong!

Most mathematicians would simply say that the three sums, S_1, S_2 and S are clearly not absolutely convergent so we are not justified in rearranging or grouping the terms in the series. However that was not good enough for several of my family so I needed a more down to earth explanation.

First note that S_1 does not equal 1/2. True the partial sums, $S_1(n)$ alternate between 0 and 1 but the infinite sum, S_1 is undefined. Now let's look at the rest of the reasoning.

In each case it's the $\ldots$ that are causing the problem. We can only ignore them if the terms captured by $\ldots$ approach zero as $n \to \infty$. We'll explain this by working with finite series of n terms and then see what happens as $n \to \infty$. Note that we can, in theory, write down all the terms in a finite series and we can rearrange the terms and sum them in any order as long as we always include all the terms.

1st explanation

Assume n is even, we get an equivalent result if n is odd but selecting one or the other simplifies the maths, to obtain the following sums with n terms

$$\begin{array}{lcllllllll} S_1(n) & = & 1 & -1 & +1 & -1 & +1 & -1 & ... & -1 \\ S_2(n) & = & 1 & -2 & +3 & -4 & +5 & -6 & ... & -n \\ S(n) & = & 1 & +2 & +3 & +4 & +5 & +6 & ... & +n \end{array}$$

Then, as before:

$$\begin{array}{lcllllllllllll} 2 \times S_2(n) & = & 1 & -2 & +3 & -4 & +5 & -6 & +... & +(n-1) & & -n & & \\ & & & +1 & -2 & +3 & -4 & +5 & +... & -(n-2) & +(n-1) & -n & & \\ & = & 1 & -1 & +1 & -1 & +1 & -1 & +... & +1 & -1 & -n & = S_1(n) - n & \end{array}$$

$$2 \times S_2(n) = S_1(n) - n \tag{4}$$

Now consider

$$\begin{array}{lcllllllll} S(n) - S_2(n) & = & 1 & +2 & +3 & +4 & +5 & +6 & ... & +n \\ & & -(1 & -2 & +3 & -4 & +5 & -6 & ... & -n) \\ & = & & 4 & & +8 & & +12 & ... & +2n \\ & = 4(& & 1 & & +2 & & +3 & ... & +n/2) \end{array}$$

So $S(n) - S_2(n) = 4 \times S(n/2) = 4 \times S(n) - 4(\frac{n+2}{2} + \frac{n+4}{2} + ... + n)$

$$3 \times S(n) = -S_2(n) + 4\left(\frac{n+2}{2} + \frac{n+4}{2} + ... + n\right) \tag{5}$$

Now lets compare the results or the finite series with the assertions in the previous section

$$
\begin{array}{rllll}
\text{from above :} & 2\times & S_2(n) & = S_1(n) & -n & (4)\\
\text{assertion :} & 2\times & S_2 & = S_1 & & (1)\\
\text{from above :} & 3\times & S(n) & = -S_2(n) & +4(\frac{n+2}{2}+\frac{n+4}{2}+...+n) & (5)\\
\text{assertion :} & 3\times & S & = -S_2 & & (2)
\end{array}
$$

We see that Equation (4) will not converge to Equation(1) as $n \to \infty$, nor will Equation (5) converge to Equation(2). In fact the differences between these, the error terms, grow to infinity. The reasoning in the assertion was wrong because it replaced these terms by . . . which were then ignored.

2nd, more rigorous and more general explanation

Define:

$$
\begin{aligned}
S_1(n) &= \sum_{i=1}^{n} (-1)^{i-1}\\
S_2(n) &= \sum_{i=1}^{n} (-1)^{i-1} i\\
S(n) &= \sum_{i=1}^{n} i
\end{aligned}
$$

Form $2 \times S_2(n)$ as above by displacing one copy by one position to the right and summing the individual terms, to obtain

$$
\begin{aligned}
2 \times S_2(n) &= \sum_{i=1}^{n} (-1)^{i-1} i + \sum_{i=2}^{n+1} (-1)^{i-2}(i-1)\\
&= 1 + \sum_{i=2}^{n} (-1)^{i-1}(i-(i-1)) + (-1)^{n-1} n\\
&= 1 + \sum_{i=2}^{n} (-1)^{i-1} + (-1)^{n-1} n = \sum_{i=1}^{n} (-1)^{i-1} + (-1)^{n-1} n\\
2 \times S_2(n) &= S_1(n) + (-1)^{n-1} n \qquad (6)
\end{aligned}
$$

Now consider

$$\begin{aligned} S(n) - S_2(n) &= \sum_{i=1}^{n} i(1 - (-1)^{i-1}) \\ &= 2\sum_{i=1}^{n} e(i)i \end{aligned}$$

Where $e(i)$ is a function I have defined to be 1 if i is even, else 0, so

$$\begin{aligned} S(n) - S_2(n) &= 4\sum_{i=1}^{[n/2]} i \\ &= 4S(n) - \sum_{i=[(n+2)/2]}^{n} i \end{aligned}$$

Where $[x]$ denotes the integral part of x. So we have

$$3S(n) = -S_2(n) + \sum_{i=[(n+2)/2]}^{n} i \tag{7}$$

All the above is rigorous and is true for all finite n.

Now we compare the finite sums developed here with the infinite sums in the assertion

from above :	$2\times$	$S_2(n)$	$= S_1(n)$	$+(-1)^{n-1}n$	(6)
assertion :	$2\times$	S_2	$= S_1$		(1)
from above :	$3\times$	$S(n)$	$= -S_2(n)$	$+ \sum_{i=[(n+2)/2]}^{n} i$	(7)
assertion :	$3\times$	S	$= -S_2$		(2)

Again showing that the finite sums cannot converge to the infinite sums stated in the assertion.

What about the connection with the Riemann zeta function

The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{8}$$

$\zeta(s)$ is defined for all, complex, values of s but the identity, Equation 8, only holds for $\Re s > 1$. This is a process called analytical continuation where the function equals the infinite sum over a certain region, the whole half planne $s > 1$, and is designed to extend smoothly over the remainder of the complex plane.

$\zeta(s)$ is, in fact, finite and well behaved at all points except at $s = 1$ where it becomes infinite. But the sum in Eq 8 is infinite for all $\Re s \leq 1$.

One particular value is $\zeta(-1) = -1/12$ which is the result the Numberphile videos quote. But this does NOT imply that

$$\sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

, since

$$\zeta(s) \neq \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{if } \Re s \leq 1$$