M840 Dissertation in mathematics

The Significance of Projective and Non-Euclidean Geometry by 1910

Mark G Watts

B8428581

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The Open University

Milton Keynes UK

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Abstract

Projective and non-Euclidean geometry developed in the nineteenth century from relative obscurity to being in the mainstream of mathematical research.

This essay tries to explain how this happened and what is the true significance of these two important branches by the early twentieth century.

To do this we trace the development of geometry from the time of the ancient Greeks up to the work of Poincaré and Hilbert in 1910. We then show how projective geometry has come to be accepted as an overall encompassing geometry with both Euclidean and non-Euclidean geometries as special cases. Also, non-Euclidean geometry has become accepted as every bit as true and valid as Euclidean geometry; as Poincaré says [41, p.104]: "One geometry cannot be more true than another, only more convenient".

This development of both projective and non-Euclidean geometries over the nineteenth century has sparked a huge research interest so that both are studied for their own sake and for the developments in other mathematical research areas that they engender, such as the theory of continuous groups. In addition non-Euclidean geometry is also studied as a potential model for the geometry of the universe.

Given the title, I have tried to write the essay as if I am writing in 1910, thus "..projective geometry **is now** regarded.." in the first paragraph of the Introduction, and similar uses of the present tense.

Contents

Chapter 1

Introduction

The short answer to the question “what was the significance of projective and non-Euclidean geometry by 1910?” is that projective geometry is now generally recognised as the single overall geometry with both Euclidean and non-Euclidean geometries being special cases obtained by projecting in different ways. Non-Euclidean geometry is considered as valid and self-consistent as is Euclidean geometry. Further, these two disciplines have brought geometry from almost a dead subject to one of lively scientific interest and much research.

However the above answer is meaningless without context and the only way to provide this is to describe some of the developments that led to the above view.

Before we even do that we need to be clear about some terms. We will use the term Euclidean geometry to describe the geometry as formulated by Euclid in *The Elements* [15]. Very specifically this includes what is generally referred to as the *parallel postulate* that, given any line and any point not in the line, there is exactly one line parallel to the given line.

We will use the term non-Euclidean geometry to refer to a geometry which rejects Euclid’s fifth postulate.

This essay will therefore start with the initial development of geometry as a science by the ancient Greeks and then trace its path up to 1910. We gloss over the period from Euclid (c 300 B.C.) to the introduction of perspective by the Renaissance artists since little of interest to this story happened during this time. We then follow two parallel paths.

One path, starting with the French mathematicians: Monge, Poncelet, Chasles, Gergonne, Cauchy, was the development of projective geometry. This was initially built on the perspective work of the Renaissance artists and all described in a descriptive manner much the same as the the way that Euclid developed geometry in *The Elements* [15] and was often called *descriptive geometry*. It was further developed through an algebraic approach by Möbius, Plücker and Cayley until Cayley was able to state “Metrical [Euclidean] geometry is thus a part of descriptive [projective] geometry and descriptive geometry is all geometry” [10, p.592].

The other path started with numerous attempts to prove Euclid’s fifth postulate and then, through the work of Saccheri, Lambert, Gauss, Bolyai (father and son) and Lobachevskii, led to the development of non-Euclidean geometry a discipline just as valid and self-consistent as Euclidean geometry. This was further extended through the work of Beltrami, Riemann, Poincaré and others until Klein built on the work of Cayley, above, and showed that non-Euclidean geometry could also be shown to be a branch of projective geometry.

Thus all branches of geometry were united as part of projective geometry.

This essay discusses the evolution of these two branches in more detail before discussing the way that geometry is viewed in 1910. This view includes the axiomatic approach coupled with the duality theories where points and lines are seen as completely interchangeable as long as the words describing connections are changed. It also includes an approach to geometry through group theory.

A further, important, aspect of the way geometry is viewed is that it has become a very active field of research and study whereas at the beginning of the nineteenth century it was largely regarded as a dead subject in which everything that could be said about it had already been said.

We also discuss the connection between geometry and the space in which we live. Is space curved or flat? However, although Einstein’s theory of *Special Relativity* and Minkowski's theory of *Space-Time* show specific examples where space may not be truly Euclidean on a local scale, there is still no indication whether our overall universe is Euclidean or non-Euclidean, nor are we likely to ever be able to find out.

Chapter 2

Geometry to End 18th Century

2.1 Greek Geometry

The geometry that is usually taught in high schools is essentially unchanged from that developed by the Greeks.

Ancient civilisations before the Greeks, especially the Babylonians and Egyptians, were already using a form of geometry. For example the Egyptians knew the properties of some Pythagorean triangles so that they could construct right angles using lengths of wood or rope. The accuracy with which this was done can be seen in the incredible precision of the pyramids constructed more than 4,500 years ago. Precise though this was it was essentially a basic construction tool rather than an academic exercise.

The first attempt to develop geometry into an academic discipline started with Thales of Miletus (c 625-547 B.C.). In the words of George Allman [2, p.7], he "introduced *abstract* geometry, the object of which is to establish precise *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous".

This geometry was further developed by many other Greek philosophers and mathematicians including Pythagoras, Plato and Eudoxus until Euclid formalised, extended and compiled all this body of knowledge circa 300 B.C. in *The Elements*, [15] [1] .

[1] Halsted gives a fascinating account of the way in which *The Elements* came, via Islamic scholars, to be eventually published in the West.

Other Geek mathematicians, especially Apollonius of Perga (born c 230 B.C.) and Archimedes (c 287-212 B.C.), further elaborated this work. Apollonius produced his *Treatise on Conic Sections* [4], quoted by the French mathematician Chasles as containing "The most interesting properties of the conics".

Among the many results that Archimedes was able to prove we single out his remarkable determination of upper and lower bounds on the ratio of a circle's circumference to its diameter, π, by an iterative method that can theoretically be continued to any degree of precision. In fact there was a very natural limit due to the need to perform all the calculations by hand and without the benefit of modern mathematical notation nor of a positional number system. Nevertheless Archimedes developed the following result : "The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$" [5, p.93], which gives π to significantly better precision than one part in one thousand..

2.2 Early ideas of Perspective and Projection

Ideas of projection were developed as the Greeks tried to map the known world. Eratosthenes (c. 275–195 B.C.) is well known for calculating the radius of the earth by measurements of the shadow projected by a vertical pole at different latitudes. Later Claudius Ptolemy (c. 90–168) introduced *perspective projection* into his maps. Mapmaking continued to evolve and reached a golden period in the 16th and 17th centuries with the work of the Flemish cartographers of whom Mercator is perhaps the best known. The *Mercator projection* that he used in his

"... The book that monkish Europe could no longer understand was then taught in Arabic by Saracen and Moor in the Universities of Bagdad and Cordova.

"To bring the light, after weary, stupid centuries, to western Christendom, an Englishman, Adelhard of Bath, journeys, to learn Arabic, through Asia Minor, through Egypt, back to Spain. Disguised as a Mohammedan student, he got into Cordova about 1120, obtained a Moorish copy of Euclid's Elements, and made a translation from the Arabic into Latin" [8, Translator's Introduction, p.1] .

1569 world map is a cylindrical projection such as one would obtain by wrapping a cylinder of paper around the earth and projecting from the axis of the earth onto the cylinder which is then unrolled into a flat piece of paper.

In parallel with this progress in cartography, artists were exploring ideas of perspective (from *perspico* — I observe) to produce more realistic paintings. This started in the early 15th century with Philippo Brunelleschi, was further developed by Piero della Francesca about 1470 and fully developed by Dürer into a mathematical theory of perspective.

A simplified form of this is shown in Figure 2.1 where we imagine viewing a horizontal scene, the plane $(x, y, 0)$, through an eye placed at $C = (0, Y_C, Z_C)$. This forms an image in the plane $(x, 0, z)$ which becomes the artists canvas. It was through use of such a mechanism that artists were able to convey a realistic representation of perspective. Some properties of this construction can be readily proved: points and straight lines in the scene are transformed into points and straight lines in the canvas. Parallel lines in the scene all meet along a line, the vanishing line, in the canvas, this is a line in the canvas at the same height as the eye, $(x, 0, Z_C)$. It is the line to which all points at infinity in the scene are mapped.

This theory of perspective was instrumental in rendering paintings more lifelike and realistic. Similarly projection was essential for producing maps of the world. But these were not considered as different forms of geometry [2] so essentially the geometry of the ancient Greeks remained the definitive form.

[2] For example Ruskin's book *Elements of Perspective*[46], a book aimed at artists, is subtitled *Intended to be read in connexion with the first three books of Euclid.*

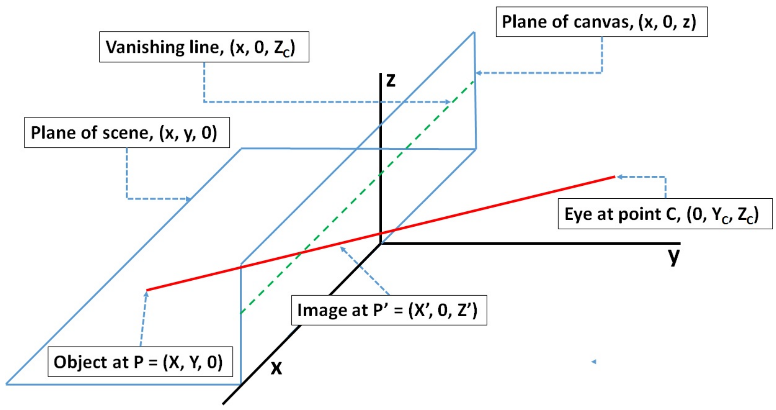

Figure 2.1: Illustrating the theory of perspective developed by Dürer. The red line shows how the point, P, in the scene is transformed into P' in the canvas

Chapter 3

Projective Geometry

3.1 Monge and the Start of Projective Geometry

Much of the early work in modern projective geometry started in France immediately after the revolution. This was partly because the new republic needed geometers to design forts and other weapons of warfare.

Monge [34] started down essentially this line and was soon a professor, teaching his methods to military students. He developed an approach known as descriptive geometry whereby a three dimensional object is represented by its projections onto orthogonal planes and properties worked out by considering each plane separately. This descriptive geometry helped to better visualise geometric objects and could greatly simplify proofs and enable less able students to develop ideas in three dimensional geometry.

As projective geometry developed following Monge, there were many claims and counter claims about who made the major contribution. We do not attempt here to follow every development or to postulate on priority. We will certainly not mention all the actors. However we believe it is instructive to follow the development of two main themes, both integral to projective geometry:- duality and cross ratio. These were both being developed at the same time and interact with each other so there is some overlap in this approach but the two themes are both central to an understanding of the status of geometry by 1910.

3.2 Duality

3.2.1 Poncelet

In 1822 Poncelet published his memoire on projective geometry, [44]. In it he develops projective geometry as a separate mathematical discipline. His concept of projection is similar to that of the artists outlined above, see Figure 2.1, but the scene and canvas are both in the same plane, as is the eye or projection point.

Poncelet uses projection from a single point to prove many theorems for conics which are relatively easy to prove in the special case of a circle. He proves the theorem for a circle then argues that projection about a suitable point can transform any conic into a circle while transforming individual points and straight lines into points and straight lines so many theorems can be simply extended from a circle to a general conic. [3] As an example Poncelet proves a duality between 'pole' and 'polar' which is illustrated here for an ellipse, Figure 3.1.

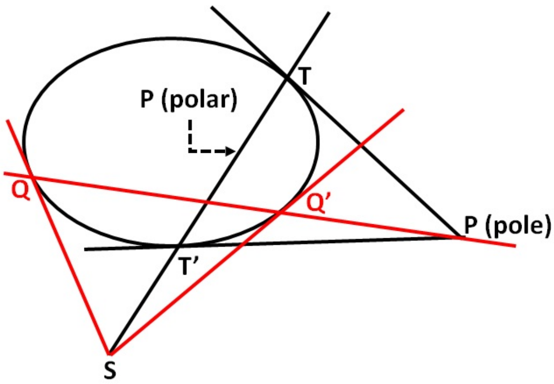

Figure 3.1: Illustrating the duality of pole and polar for an ellipse

For any ellipse we choose a point outside the ellipse and call this point, P, a pole. Construct two tangents from P to touch the ellipse at points T and T'. The line

[3]This is especially true of theorems involving multiple points in a straight line or multiple lines intersecting at a single point. It does not apply to any theorems concerning the magnitude of a distance or an angle

TT' when extended in both directions is called the polar, p. Now chose any point S on p outside the ellipse, call it a new pole and create a new polar, QQ', by constructing the tangents SQ and SQ'. QQ' will pass through the original pole, P. Thus there is a duality between pole and polar.

This is proved for the circle in Appendix A and can then be extended to all ellipses with the pole outside the ellipse by the simple projection defined above. Extension to other conics and to cases where the pole is inside the conic, or otherwise situated such that it is impossible to construct two real tangents to the conic, are dealt with by Poncelet as he introduces concepts such as *ideal chords* and his *principle of continuity*. This is significantly more complicated, maybe even vague, and its justification is still open to some controversy.

A book such as Poncelet's required approval from the French Académie Royale des Sciences and Poncelet has included the report which was prepared by Poisson, Arago and Cauchy. Cauchy chaired the commission and was the most eminent mathematician of the three reviewers (probably the most eminent in France) so we can assume that this mainly expresses his ideas. While the report recommends publication, it does express concern with parts and Cauchy encourages a more algebraic approach including using complex numbers as a way of removing the difficulties with *ideal chords* and the *principle of continuity*. Thus Cauchy started to move projective geometry towards a more algebraic approach and may even have anticipated the later work of Plücker and Cayley.

3.2.2 Gergonne

Gergonne [19] further extended Poncelet's concepts to demonstrate that a complete duality exists between *points* and *lines* provided that the words used to denote connections — intersect, lie on, cross, meet etc — are adjusted accordingly. He cleverly laid out his results in two columns, one referring to lines connecting points and the other referring to points as the intersection of lines. In this way he was able to (almost) point out the complete duality between points and lines. This is a much deeper insight than that between pole and polar and later became the basis of the modern axiomatic approach to projective geometry.

The word “almost” was introduced above because Gergonnes work suffered from two flaws. The first was quickly rectified by Gergonne himself but the second required the algebraic approach of Plücker to resolve.

When we consider the dual of a curve we must consider the pole and polar. Thus if a curve has n tangents from a given point, the pole, its dual will cross the resulting polar at n points. Gergonne introduced the terms degree and class defined as follows. The *degree* of the curve is the maximum number of points at which it is intersected by a straight line. The *class* of the curve is the maximum number of tangents to the curve from a fixed point.

For any curve of degree n the class equals $n(n-1)$. Conics are of degree 2 so also of class 2.

Gergonne’s first flaw was that he did not originally distinguish degree from class. He used the same term, *order*, for both since he was clearly thinking in terms of conics where degree and class are both equal to 2. This was quickly corrected simply by introducing the terms degree and class. The second flaw was more serious and again it only showed up for curves of degree higher than 2.

3.2.3 Plücker

One of the beauties of duality is that if it is applied twice to a curve the resultant curve should be the same as the original. However consider forming the dual of the cubic, a curve of degree 3 so class $3(3-1)=6$. The class is the maximum number of tangents from a fixed point so, when we form the dual, this becomes the maximum number of points in which the curve is intersected by a straight line. Therefore the degree of the dual should be 6. Worse follows! The class of the dual should now be $6.5=30$. If we now form the dual of this dual its degree should therefore be $6(6-1)=30$. We have now applied duality twice to a curve of degree

3 and obtained, instead of the original curve, a curve of degree 30!

This anomaly was resolved by Plücker [38] by showing that multiple points and cusps have to be treated especially carefully when passing over to the dual image. Multiple points and cusps are called singular points. The dual of a double point (point where the curve crosses itself) is a double tangent (line which touches the curve at two points). The dual of a cusp is a point of inflection. Plücker showed that the dual of a curve of degree n with d double points and c cusps is a curve of degree $n(n-1) - 2d - 3c$. With this correction the dual of the dual of a cubic is again a cubic. The same is true for higher degree curves.

Salmon wrote two excellent textbooks in 1848 and 1852 which formalise and combine all this earlier work, and more, using an algebraic approach with homogeneous coordinates throughout. One deals primarily with conic sections [49] and the other with curves of higher degree, the higher plane curves [50].

3.2.4 Möbius

Whereas Monge, Poncelet, Gegonne and other, mainly French, mathematicians had used a descriptive approach to projective geometry, Plücker used an algebraic approach based on the equations of the curves. This approach was already advocated by Cauchy in his review of Poncelet's memoire [44] but it was first taken up seriously by Möbius. He introduced a concept called *barycentric coordinates*, which he later simplified to projective, or homogeneous, coordinates, where each point in a plane was identified by three coordinates rather than the usual two — only the ratios mattered. This enabled him to identify the point at infinity in a natural way and not as some limiting process. The point at infinity therefore became a specific point in the projective plane so statements about lines and points became simpler and more symmetric. For example any two points define a line, any two lines meet in a single point. We no longer need to make an exception in the case of parallel lines because their meeting point is the point at infinity which has become a perfectly valid point.

Homogeneous coordinates are defined as follows. Consider the equation of a curve in a plane. This will be a function of powers of x and y. Replace each instance of x by x/z and each instance of y by y/z and multiply through to remove inverse powers of z. The result will be a homogeneous equation in the three variables x, y and z. As an example consider the general quadratic equation

$$ax^2 + by^2 + cxy + dx + ey + 1 = 0$$

Introducing homogenous coordinates this becomes the homogeneous equation

$$ax^2 + by^2 + cxy + dxz + eyz + z^2 = 0$$

3.3 Cross Ratio

3.3.1 Geometrical Invariants

Veblen and Young [53, §1] start their book on projective geometry with the statement

"Geometry deals with the properties of figures in space. Every such figure is made up of various elements (points, lines, curves, planes, surfaces etc.), and these elements bear certain relations to each other (a point lies on a line, a line passes through a point, two planes intersect etc.). The propositions stating these properties are logically interdependent, and it is the object of geometry to discover such propositions and to exhibit their logical interdependence."

Some of the properties Veblen and Young refer to are invariants:- they remain constant under certain transformations. In the two dimensional geometry of Euclid the possible transformations of the plane are rotation about a fixed point and translation. Under these transformations points are mapped to points and straight lines are mapped to straight lines. Further, Euclidean distance, defined by applying Pythagoras' theorem, is conserved. Because distance is conserved we can draw identical triangles in both the un-transformed and transformed systems so that angles must also be conserved.

In projective geometry we define a perspectivity as a single projection about a given point as shown in Figure 3.2. A projectivity is then defined as the product of one or more perspectivities.

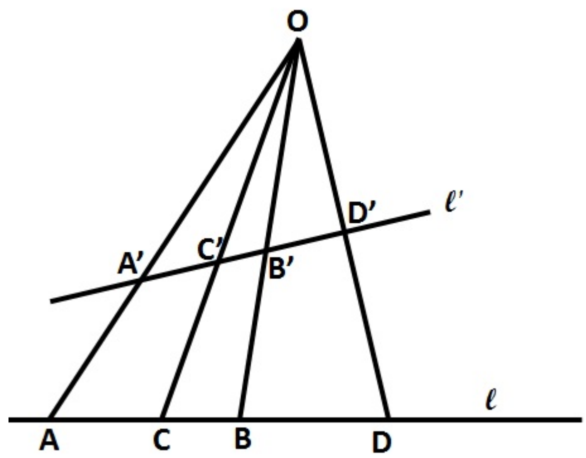

Figure 3.2: Illustrating a perspective transformation or perspectivity. Points A, C, B, D on line l are mapped to A', C', B', D' on line l' via projection in O.

A projectivity will send straight lines to distinct straight lines and points to distinct points. It will retain the order of the points on a line in the sense that if C is between A and B then C' will be between A' and B'. However it will not preserve lengths or angles. Lengths and angles are not invariants in projective geometry.

3.3.2 Cross Ratio Defined

A further conserved property in projective geometry is the cross ratio. We will introduce this by first defining the harmonic ratio, following Poncelet [44, p.12]

Referring to Figure 3.2 we say that C and D cut the line AB harmonically if [4]

$$\frac{AC}{BC} = -\frac{AD}{BD}$$

and it also follows that A and B cut the line CD harmonically. A little rearrangement gives

$$\frac{2}{CD} = \frac{1}{AD} + \frac{1}{BD}$$

So that CD is the harmonic mean of AD and BD.

Still referring to Figure 3.2 we define the anharmonic ratio, or cross ratio, of the 4 points A, C, B, D as

$$(A, B; C, D) = \frac{AC/BC}{AD/BD} = \frac{AC \times BD}{AD \times BC}$$

So that the cross ratio equals -1 if C and D cut the line AB harmonically. This is the reason the cross ratio was originally called the anharmonic ratio: it measures the deviation from a harmonic arrangement of four points on a line.

It is easily proven that any perspectivity preserves the cross ratio of any four distinct points on a line. Therefore any projectivity, being the product of one or more perspectivities, will also preserve this ratio.

Poncelet states that Pappus of Alexandria was already aware of some properties of cross ratio but attributes to Brianchon (1807) the proof that cross ratio is invariant under a perspectivity.

3.3.3 Chasles and Cross Ratios Defined by Angles

Chasles [11, p.11] showed that we can also define cross ratio in terms of the angles subtended at the point of perspectivity. Thus, still referring to Figure 3.2, he

[4] I have changed the terminology slightly since Poncelet appears to refer to the length of the line from A to C (to the right of A) as CA. I have used AC for this length to be consistent with other authors, however the result is unchanged since all the negative signs appear in pairs. He does, however, appear to have missed a negative sign, I believe because he is thinking of a different arrangement of points from that shown in his Figure [44, Fig. 2], which is essentially the same as Figure 3.2, above.

proved [5]

$$\frac{\sin(AOC)}{\sin(AOD)} \div \frac{\sin(BOC)}{\sin(BOD)} = \frac{AC}{AD} \div \frac{BC}{BD}$$

This equation immediately proves that cross ratio is preserved for all projections about a point onto any straight line. It also shows that cross ratio of angles is preserved as the point of projection is moved, with the four points held constant on a line. Thus this relation is another example of duality.

3.3.4 Cayley and the *Absolute*, Link to Euclidean Geometry

Projective geometry is normally described as non-metrical because distances and angles are not preserved under projection. However Cayley [10] argued that cross ratio can be used to define a measure of length if two of the points are fixed. The two points are the points of intersection with a certain conic called the *Absolute*.

"This absolute configuration must clearly be a curve which every straight line cuts in two points, real or imaginary, and to which two tangents can be drawn from every point: i.e. it must be a conic. The absolute in the plane is therefore a fixed conic, ... in the geometry of Euclid the *absolute* must be a degenerate conic consisting of a pair of points, viz. the circular points at infinity" [17, p.157] [51].

With this definition of the *absolute* Cayley was able to state [10, p.592]

"Metrical [Euclidean] geometry is thus part of descriptive [projective] geometry and descriptive geometry is all geometry".

Thus Cayley had shown that Euclidean geometry was a special case of projective geometry. As we discuss later Klein then expanded on this to show that non-Euclidean geometry is also a special case of projective geometry. and produced what is know referred to as the Cayley-Klein metric.

[5] Chasles uses lower case symbols for points and upper case for lines. I have changed this for consistency

Chapter 4

Non-Euclidean Geometry

4.1 Euclid's Postulate

Euclid's *The Elements* [15] covers thirteen books containing all of the arithmetic and geometry known by about 300 B.C., for example it contains his famous number theory proof that there are infinitely many prime numbers (Book IX, Proposition 20).

However the majority concerns geometry and it starts in Book 1 with *definitions*, *postulates* and *common notations*.

We will not quote all 23 definitions but the last is of particular importance to us "parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction".

The *common notations* are 5 supposedly self evident truths such as "Things which are equal to the same thing are also equal to one another".

There are 5 postulates which are quoted in full below:

1. To draw a line from any point to any point
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.

5. If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

Postulates 1 and 2 would today be written something like

1. Any two non-coincident points uniquely define a straight line.

2. A straight line can be continued indefinitely without intersecting itself.

Postulate number 5, the parallel postulate, has been the subject of much debate ever since *The Elements* were first written. There is a strong suggestion that Euclid was not sure of the validity of this postulate since he placed it last and ordered his propositions (theorems) such that early ones did not depend on the parallel postulate. It is only in proposition 29 that the parallel postulate is first used. The parallel postulate has been stated in many different, but essentially equivalent, ways. One form is what is generally known as Playfair's axiom: "given any line and a point in the same plane but not on the line, there is exactly one line through the point which is parallel to the given line" [6].

There are many consequences of the parallel postulate which are often taken for granted, such as that the interior angles of any triangle sum to two right angles or 180°. Also that triangles can be scaled and still retain the same angles — similar triangles.

Many mathematicians, starting with the ancient Greeks:- Diodorus, Proclus and Ptolemy; through the Persians to the great French mathematician, Legendre (1752-1833), have tried to prove the parallel postulate from the first 4 postulates but to no avail. A good summary of these accounts is given in Euclid's *The Elements*, Book 1, *Note on Postulate 5* [15, pp.202-220]. All these attempts to

[6] In fact Playfair writes this as "That two straight lines which cut one another cannot be both parallel to the same straight line" and attributes this statement to Ludlam [37, p.291] .

prove the parallel postulate show how deep seated was the common belief in its truth.

4.2 Saccheri 1667-1733

Saccheri [48] took a somewhat different approach, aiming for a proof by contradiction, *reductio ad absurdum.* He was also totally convinced of the truth of the parallel postulate so he set out to develop a self-consistent geometry based on the first four postulates but denying the fifth. The intent was that he would at some point find an inconsistency which would automatically justify the parallel postulate.

In 1733 he published his *Euclides ab omni naevo vindicatus*, [*Euclid freed of every fleck*], subtitled *a geometric endeavor in which are established the foundation principles of universal geometry.*

Saccheri starts by assuming the first 4 postulates and refuting the fifth. He then develops very simple geometrical arguments based on the simple construction, Figure 4.1, of a straight line, AB, on which he raises 2 equal perpendiculars, AC and BD, and joins CD. Use of the parallel postulate would show this to be a rectangle so $CD = AB$ but Saccheri does not assume this. Instead he shows that the angles at C and D must be equal and distinguishes 3 cases [7]:

har: right angle, $C = D = \pi/2$, $CD = AB$
hao: obtuse angle, $C = D > \pi/2$, $CD < AB$
haa: acute angle, $C = D < \pi/2$, $CD > AB$

He then goes on to show that case *har* implies that all triangles will have angles summing to π, *hao* implies they will always sum to values $> \pi$ and *haa* that they will always be $< \pi$. He is then able to show that case *hao* produces a contradiction

[7] I have used the abbreviations of the names Saccheri gives to each of these hypotheses, respectively: hypothesim anguli recti, hypothesim anguli obtusi, hypothesim anguli acuti

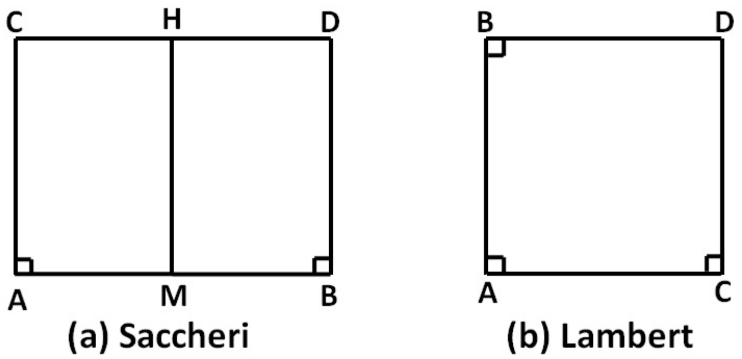

Figure 4.1: The quadrilaterals constructed by Saccheri (left) and Lambert (right) with the right angles marked. Saccheri inserts the line MH where M and H are the mid points of AB and CD respectively and then proves that angles AMH and CHM are right angles so Lambert's quadrilateral is exactly the same as the right half of Saccheri's quadrilateral

and, erroneously, proves that *haa* also has a contradiction. That he gets to this latter result despite all the good earlier work is probably due to his total belief in the parallel postulate. He needed to prove it true so was somewhat blind to the error in his arguments. This is somewhat equivalent to the well known concept of "flinching" in experimental work where the experimentalist knows the result he expects and keeps measuring until he arrives at it.

Saccheri talks of wanting to tear out by its very roots "the hostile hypothesis of acute angle". Halsted comments [21, p.101] "Saccheri was always fighting against the heretical results of his own logic on behalf of what he considered God's truth." In fact Saccheri had gone a long way towards discovering hyperbolic geometry, over 100 years before Lobachevskii's famous paper which will be described later.

4.3 Lambert 1728-1777

It is not clear how much Lambert knew of Saccheri's work. He starts from a different position with a diagramme similar to that later used by Lobachevskii, see Figure 4.2 but then in section 39 [14, p.180] introduces a quadrilateral very similar to that used by Saccheri Figure 4.1 and lists 3 hypotheses, he will develop, namely

Hypothesis 1. $BDC = 90$ degrees

Hypothesis 2. $BDC > 90$ degrees

Hypothesis 3. $BDC < 90$ degrees

So this part of his work is very similar to Saccheri's.

He shows, as had Saccheri, that hypothesis 2 leads to a contradiction if we retain Euclid's second postulate that lines can be continued indefinitely without meeting. He failed to prove that his hypothesis 3 (acute angle) led to a contradiction and thereby developed a new self-consistent geometry, now known as hyperbolic geometry.

Lambert also showed that a self-consistent geometry can be developed based on the hypothesis of an obtuse angle, hypothesis 2, if he relaxed Euclid's second postulate and allowed the radius to be imaginary. This is similar to the geometry on the surface of a sphere and is now called spherical geometry.

Lambert was able to prove that in both hypotheses 2 and 3 the angles of a triangle did not sum to two right angles. If the angles are denoted by α, β and γ he proved that $\alpha + \beta + \gamma - 2\pi$ is positive for hypothesis 2 (obtuse angle) and negative for hypothesis 3 (acute angle) and, in both cases, is proportional to the area of the triangle. This then led him to the conclusion that there must be an absolute measure of length since there is obviously an absolute measure of angle.

It appears that Lambert was not satisfied with this work, as he had set out, and failed, to find an inconsistency which would validate Euclid's fifth postulate, and he did not publish it in his lifetime. Frankland [16] suggests that the work was completed in 1766. It was published posthumously in 1786 as "Die Theorie der Parallellinien" and was then republished by Engel and Stäckel [14] in 1895 [8]

Lambert makes the following comment on his failure to disprove his 3rd hypothesis [16, p.25]:

"This consequence possesses a charm which makes one desire that the Third Hypothesis be indeed true!

"Yet on the whole I would not wish it true, notwithstanding this advantage (of an absolute standard length), since innumerable difficulties would be involved therewith. Our trigonometric tables would become immeasurably vast; the similitude and proportionality of geometrical figures would wholly disappear, so that no figure can be represented except in its actual size; astronomy would be harassed."

4.4 Lobachevskii

Lobachevskii is today generally regarded, along with Gauss and Bolyai, as being the founder of hyperbolic geometry. His translator, Halsted, tells us that [29, p.8] "the first public expression ... dates back to a discourse at Kasan on February 12, 1826". His paper was then published, in sections, in Russian, in the *Kasan Messenger*, between 1829 and 1830 [20, p.120], and totally ignored outside of Russia. Lobachevskii then translated this into German and republished it in a German journal in 1840 and this is the version that is now most known.

Lobachevskii's starting point is his definition of *parallel* and *angle of parallelism*, Figure 4.2. He splits lines through A into 2 classes, those cutting BC and those not

[8] Despite an extensive literature search I have been unable to find an English translation of Lambert's paper. This is a great shame since it was originally written in 1766, fully 60 years before Lobachevskii first started to share his views on hyperbolic geometry, and therefore should be considered the first publication on non-Euclidean geometry.

cutting, and calls the boundary line, AH, between the 2 classes the *parallel line.* [9]

The angle of parallelism, $\Pi(p)$, is defined as the angle between AH and AD, the perpendicular to BC. $\Pi(p)$ is a function of p, the perpendicular distance of A from BC. If $\Pi(p) = \pi/2$ we clearly have Euclidean geometry. Lobachevskii focuses on the case where $\Pi(p) < \pi/2$.

It is important to notice that, if we define line AH to be parallel to BC, then the continuation of HA beyond A will not, in general, be parallel to BC.

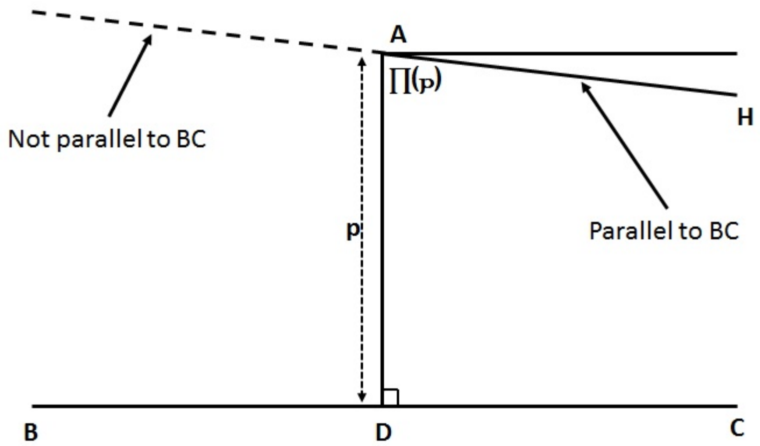

Figure 4.2: Definition of parallel and the angle of parallelism, $\Pi(p)$

He defines a *horocycle* as a curve such that the perpendicular bisectors of all its chords are parallel, see Figure 4.3. Euclidean geometry holds on a horocycle so he is able to show that the length of a horocycle between two parallel lines reduces as e^{-x} where x is the distance along an axis between the two horocycles. See Figure 4.4 which gives $s' = se^{-x}$.

From the relation $s' = se^{-x}$, Lobachevskii is able to derive the relationship

$$\tan \frac{1}{2}\Pi(x) = e^{-x} \tag{4.1}$$

[9]Lobachevskii makes a distinction between "parallel" and "not cutting" which had not been made before. In fact these terms were almost used interchangeably in Euclidean geometry.

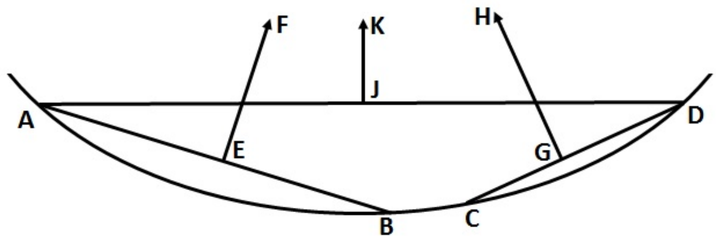

Figure 4.3: Illustrating the definition of a horocycle. EF, GH and JK are parallel in the definition of Lobachevskii

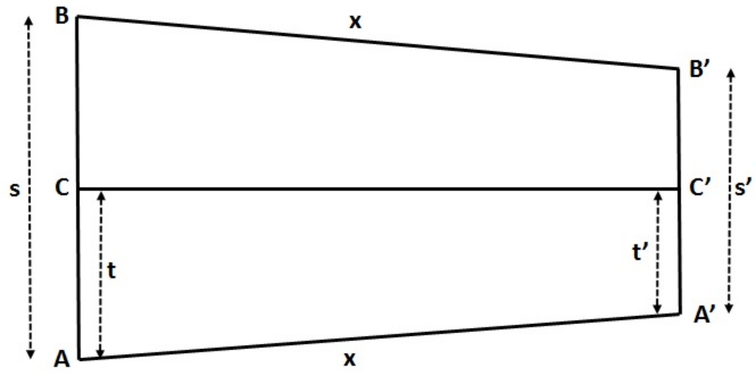

Figure 4.4: Illustrating reduction of a horocycle between parallel lines, $s' = se^{-x}$

And he was then able to develop relationships between the angles of a triangle, valid in hyperbolic geometry [10], which reduce, for small triangles, to the sine rule, the cosine rule and the condition that the angles of a triangle sum to 180°.

He also shows that his equations reduce to identities pertaining to spherical triangles if lengths are all multiplied by $\sqrt{-1}$ [11]. This is similar to the result obtained by Lambert.

Lobachevskii's fundamental relationship which was shown as Equation 4.1 directly

[10] These relationships include $\Pi(x)$, the angle of parallelism, so are dependent on the size of the triangle. This explains Lambert's concern over the size of trigonometric tables, see Section 4.3

[11] This will replace the hyperbolic functions by trigonometrical functions.

relates an angle to a distance and so this again shows that hyperbolic geometry directly implies an absolute measure of distance.

4.5 Gauss

Gauss is now credited, along with Lobachevskii and Jànos Bolyai, with the discovery of non-Euclidean geometry although he published very little during his lifetime. His unpublished work, discovered after his death, show that his ideas were very well developed.

While he was working as a geodesist, directing the triangulation survey of Hannover, Gauss developed a measure of the curvature of a plane surface. He defined *Gaussian curvature* as $\frac{1}{r_1 r_2}$ where r_1 and r_2 are the greatest and least radii of curvature at a point on a surface. In this he is relating his curved surface to a third, orthogonal, dimension since he was actually interested in the curvature of points on the surface of the earth.

From Gauss's definition of curvature we see that the curvature on both the inside and outside surface of a sphere must be positive, that on the surface of a cylinder must be zero. Negative curvature will be obtained near a saddle point where the curvature in one plane is positive and in the orthogonal plane negative.

To derive his expression for curvature, Gauss assumes that the position on a curved surface, (x, y, z) can be expressed as a function of two variables, u, v [12] and writes a line element in the form

$$ds = \sqrt{Edu^2 + 2Fdudv + Gdv^2}$$

where E, F and G are sums of products of the partial derivatives of the x, y and z coordinates, with u and v. He then writes the curvature, k, as an expression involving just E, F, G and their first and second derivatives with respect to u and

[12] Gauss uses p, q but I use u, v here for consistency with later authors such as Beltrami, Section 4.8.

v. [18, p.20]

"The analysis developed in the preceding article thus shows that for finding the measure of curvature there is no need for finite formulae, which express the coordinates x, y, z as functions of the indeterminates u, v; but that the general expression for the magnitude of any linear element is sufficient." — As long as you can find E, F, G as functions of u and v you can calculate the curvature.

Significantly, Gauss had shown that, although he had introduced a third dimension, the equation for the curvature does not involve this third dimension. This was later expanded by Riemann who developed a general theory of curvature of manifolds.

Gauss showed that the curvature of a triangle on a curved surface can be obtained simply by summing its angles "The excess of the sum of the angles of a triangle formed by shortest lines over two right angles is equal to the total curvature of the triangle." [18, p.48] [13].

He illustrated this theory via direct measurements between 3 hilltops in Germany, one side of was over 100 km, where the excess in the sum of the angles over two right angles is 14.85 seconds of arc.

4.6 Bolyai

Jànos Bolyai developed a theory of parallels [8] independently from, but remarkably similar to, that of Lobachevskii. It was originally published in 1832 as an appendix to a joint work with his father, Farkas (Wolfgang) Bolyai. This in turn contains an appendix to the appendix, written by Farkas, which points out many of the similarities between his son's work and that of Lobachevskii. For example, Bolyai introduces a concept called an L-curve, similar to Lobachevskii's

[13] Naturally this must be normalised for the size of the triangle, Gauss explains how the units should be chosen to achieve this.

horocycle, and an F-surface, similar to the horosphere, which is a surface of revolution of the L-curve. The useful property of the L-curves (and of horocycles) is that Euclidean geometry holds on them and therefore properties in hyperbolic geometry can be related to trigonometric functions.

Since Bolyai's conclusions are similar to Lobachevskii's we will not discuss them further here but one significant difference in their approaches is that Lobachevskii uses a descriptive approach, in which he derives theorems in a manner similar to the approach taken by Euclid, whereas Bolyai uses an algebraic approach and develops many of his results through application of calculus.

4.7 Riemann

One of Riemann's most celebrated works in geometry is his *Habilitationsvorstrag* [45], the lecture with which he defended his doctoral thesis, presented in front of Gauss and a full lecture theatre of academics and students. This paper contains only one equation so he has to express all his profound ideas in words. It develops a totally new approach to geometry based on the concepts of multiply extended quantities and of manifolds. The concept of multiply extended quantities was used to build up the notion of an n dimensional space without having to resort to Cartesian or any other coordinate system. The concept of a manifold described a portion of space which could have a totally different structure, and curvature, from its neighbouring manifolds, except, of course, that the whole of space must be continuous.

In this lecture, Riemann developed a theory of a generalised space which could have any curvature but without mentioning hyperbolic geometry. Several of his audience were from the Philosophy department, of which Mathematics was a part, and did not need to be shocked by such outrageous ideas as non-Euclidean geometry. Nevertheless Riemann specifically develops his ideas of non-zero curvature. He considers both positive and negative curvature and points out that geometries with positive curvature can be wrapped around the surface of a type of sphere and so are finite provided Euclid's second is also rejected. This became known as Riemannian or spherical geometry[14]

He also shows that negative curvature is just as valid as is zero curvature so laying the foundation for the legitimisation of hyperbolic geometry.

Riemann's single equation in his *Habilitationsvorstrag* is an expression relating curvature to the length of a line element. He starts from, but does not state in this lecture, the assumption that the line element can be written in the form [15]

$$ds = \sqrt{\sum a_{ij} dx_i dx_j} \tag{4.2}$$

He then states that, if we write α for the curvature, this expression for the line element becomes

$$ds = \frac{1}{1 + \frac{\alpha}{4} \sum x_i^2} \sqrt{\sum dx_i^2} \tag{4.3}$$

Or, writing $\frac{1}{r^2}$ for the curvature, α, we obtain in two dimensions

$$ds = \frac{\sqrt{dx^2 + dy^2}}{1 + (1/4r^2)(x^2 + y^2)} \tag{4.4}$$

the form in which it is quoted by Gray [20, p.200].

Helmholtz [22, p.48] explores the validity of the assumptions contained in Equation 4.2. He makes 4 assumptions:-

[14] This is not exactly the same as the geometry of the surface of the earth since geodesics meet in two points on the earth (e.g. all lines of longitude meet at the North and South poles). This would also be a violation of Euclid's first postulate which Riemann retains.

[15] Klein [27, p.86] clarifies this by saying "...he only wants to say that there exists an invariant of the 'form' $\sum a_{ij} dx_i dx_j$; he does not mean to imply that the three-dimensional space necessarily exists as a curved space in a space of four dimensions."

- Continuity
- Rigid bodies can move freely (implies constant curvature)
- Free mobility of points
- Invariance under rotation (called *mondromy* by Helmholtz)

and then states "it follows from pure analysis that a homogeneous function of the second degree of the magnitudes du, dv, dw exists..." . With this he shows that Equation 4.2 is the simplest general form satisfying his 4 assumptions.

Riemann had introduced a new concept in geometry, to quote Gray [20, p.201] "To him, geometry was to do with concepts like length and angle which could be intrinsically defined on a surface or space of some sort. It follows that there are many geometries, one for each kind of surface and each definition of distance...".

Thus Riemann had broadened the whole of geometry and in the process had given his stamp of approval, as the leading mathematician of his generation, to non-Euclidean geometry as a valid geometry.

4.8 Beltrami and Models of Hyperbolic Geometry

In two remarkable papers in 1868 [6] [7], Beltrami further developed Riemann's ideas about curvature and in so doing developed relationships between hyperbolic geometry and models which can be visualised in a Euclidean disc or sphere. Thus he was the first to develop models of hyperbolic geometry. He developed all this in terms of transformations of coordinates so there are no diagrammes to show the nature of the underlying projections.

In his first paper, usually referred to as his *Saggio* [6], essay, he develops the mathematics of what is know known as the Klein disc, see Section B.1. In the second paper [7] he develops several other models including the conformal model which is now referred to as the Poincaré disc, Section B.2.

This work was largely ignored at the time of publication and both the above and other models were later rediscovered or further developed by others and Beltrami's name has largely been forgotten as the originator of these models.

They are described in more detail in Appendix B.

4.9 Klein

Klein realised that Cayley's unification of projective and Euclidean geometries, see Section 3.3.4, could be further extended to unite non-Euclidean geometry with projective geometry. This can be done by choosing a different *absolute* but Klein proceeds from first principles by first defining how to set up a measure of length, then defining a scale which is exponential and defining the length as a logarithmic measure on this scale. Thus he defines the distance between two elements as

$$c \log \frac{z}{z'}$$

and interprets z/z' as a cross ratio when two other points on the line have been moved to $z = 0$ and $z = \infty$, i.e. $\log z = \mp\infty$.

This works for real one dimensional lines where the line passes through the point $z = 0$. In two dimensions Klein simply uses complex lines to describe lengths and obtains a result similar to that of Cayley.

Kleins use of cross ratio and the Cayley-Klein metric are illustrated in a diagramme of the Klein disc in Section B.1.

4.10 Poincaré

Poincaré was a true polymath and made significant contributions in many fields of mathematics, physics, engineering and philosophy. He was able to link supposedly unconnected fields. Thus his work on what he called fuschian groups [40] led to his independent development of the Poincaré disc model, see Section B.2.

Fuschian groups are a class of elliptic functions (doubly periodic functions) which are invariant under a group of mappings of the form

$$z \mapsto \frac{az+b}{cz+d}, \qquad \text{where } ad - bc \neq 0 \tag{4.5}$$

This type of bilinear transformation is called a *Möbius transformation* and Poincaré realised that it had the following characteristics

- It preserves angles
- It is monogenic (one to one)
- It maps circles into circles
- "Finally [39, p.124], if z_1, z_2, z_3, z_4 are four values of z and if t_1, t_2, t_3, t_4 are the corresponding values of t, then

$$\frac{t_1 - t_2}{t_1 - t_3} \quad \frac{t_4 - t_3}{t_4 - t_2} = \frac{z_1 - z_2}{z_1 - z_3} \quad \frac{z_4 - z_3}{z_4 - z_1}\text{''}$$

Therefore this was an ideal model of a conformal (angle preserving) mapping. In particular the last property shows that it preserves cross ratio so it is also a projective mapping. This then became the basis of Poincaré's development of the disc that bears his name. If we set $|a/c| = 1$ this maps the entire plane into the open unit circle. Poincaré also showed that depending whether the value of $(a+d)^2$ is less than, equal to, or greater than 4 this can be a model of spherical, Euclidean, or hyperbolic geometries respectively.

In the above paper Poincaré focussed on real transformations where a, b, c, d are all real. In a subsequent paper entitled *Memoire on Kleinian Groups* [40] he considers a, b, c, d all complex in Equation 4.5. He shows that the Möbius transformation that defines this mapping can also be arrived at by an even number of inversions in the unit disc and finally is able to show a connection between the theory of linear transformations and non-Euclidean geometry.

Thus through the work of Klein, Beltrami and Poincaré a total connectivity had been made between projective and non-Euclidean geometries and therefore also Euclidean geometry.

‘

Chapter 5

State of Geometry by Early 20th Century

5.1 Projective Geometry

Although projective geometry is still often taught as an extension of high school geometry, the formal, axiomatic, approach is now usually favoured by mathematicians.

A central part of this is *duality*, based on the concepts introduced by Gergonne but further developed. No distinction is made between points and lines, or even planes. They are simply abstract concepts that are connected by certain relationships. Veblen and Young [53, p.29] call this connectivity *on* so that points are said to lie *on* a line, lines lie *on* a point etc. They then state:

“Any proposition ... concerning the points and lines of a plane remains valid, if stated in the *on* terminology, when the words *point* and *line* are interchanged.

“...Any proposition ... concerning the planes and lines through a point remains valid, if stated in the *on* terminology, when the words *plane* and *line* are interchanged.”

Another essential feature of projective geometry is the algebraic approach and the use of homogeneous coordinates where x and y are replaced by x/z and y/z, see Section 3.2.4. This enables significant simplifications, including:

- The point at infinity appears naturally by setting $z = 0$, thus the projective plane has an extra point. There is now no need to have exceptions to rules as in for example *any two lines meet at a point* ***unless they are parallel***

- The point at infinity leads to the distinction between *ordinary points*, where $z \neq 0$, and *Ideal points*, where $z = 0$. This definition of *Ideal points* opens the way for the work of Cayley and Klein on unifying projective geometry with Euclidean and non-Euclidean geometry

- Equations of points, lines and planes appear in a more symmetrical form. Thus a point can be denoted (a, b, c), meaning $x = a, y = b, z = c$, and a line can be denoted by (ξ, η, ζ) meaning $\xi x + \eta y + \zeta z = 0$. Further the requirement that the point is *on* the line is $a\xi + b\eta + c\zeta = 0$ which is also the requirement that the line is *on* the point.

- Many relationships and transformations, e.g. between points and lines, can be expressed via linear algebra and can therefore be written in matrix notation.

Modern geometry also began to be stated in terms of group theory through the work of Sophus Lie but here explained by Poincaré [42, p.9]:

"It is obvious that if we consider a change A, and cause it to be followed by another change B, we are at liberty to regard the ensemble of the two changes A followed by B as a single change which may be written $A + B$ and may be called the resultant change. (It goes without saying that $A + B$ is not necessarily identical with $B + A$.) The conclusion is then stated that if the two changes A and B are displacements, the change $A + B$ also is a displacement. Mathematicians express this by saying that the ensemble, or aggregate, of displacements is a group. If such were not the case there would be no geometry." [16]

Group theory implies symmetries and invariants. Thus two invariants of Euclidean

[16]Nowadays we would write $B.A$ where Poincaré writes $A + B$.

geometry are distance defined by

$$ds^2 = dx^2 + dy^2$$

and the size of angles. One of the invariants of projective geometry is the cross ratio which Cayley and Klein had shown to yield distance and angle through the correct choice of the *absolute*.

Klein published his *Erlangen Programme* [26] in 1872. In it he sets out his view of all the most important problems that need to be worked on in projective geometry. In effect a research proposal valid for many years. These were all areas that he found particularly interesting and it is notable that the main area he focusses on is the various group theoretical approaches to geometry. He refers to the "group of space-transformations" and defines the "principle group" as the group of all "space-transformations by which the geometrical properties of configurations in space remain entirely unchanged" [26, §1]. He was at the time working with Lie so this obviously represented a very fruitful set of research areas for both of them. In fact Klein was convinced that group theory was central to geometry and that geometry should be the study of those elements which are invariant under certain groups of transformations.

5.2 Euclidean Geometry

Naturally, Euclidean geometry is still taught at the high school level as it is much easier conceptually and already contains all the tools needed for an understanding of the physical world as we experience it. It also teaches a disciplined and rigorous approach to problem solving.

The standard textbook is either a direct translation of The Elements or a translation with commentary and explanation such as Playfair [37].

If one accepts the parallel postulate there has still been a problem with Euclidean geometry until recently. The problem was the axioms as stated by Euclid.

Mathematicians had for a long time agreed that the approach used by Euclid and the definitions and axioms he specified were not rigorous enough. Heath refers to disagreements between Plato and Aristotle over the definition of a *point* [15, p.155] and this was even before the time of Euclid. This and similar disagreements did not stop.

Recently Hilbert published his *Foundations of Geometry* [24] where he takes a rigorous, axiomatic, approach to geometry but not by treating it as a subset of projective geometry. Hilbert introduces a set of axioms of:- connection, order, parallels, congruence and continuity; and shows how geometry can be rigorously developed based on these axioms. He shows how the axioms are independent of each other and how different, but equally valid, geometries can be defined by rejecting certain axioms.

With this work Euclidean geometry has been made as rigorous as projective geometry.

5.3 Non-Euclidean Geometry

Non-Euclidean geometry is now recognised as a completely rigorous discipline. Through the approach of Klein it was shown to be a special case of projective geometry, as is Euclidean geometry. Therefore non-Euclidean geometry is every bit as complete and rigorous as Euclidean geometry. The writings of Riemann and Poincaré have done much to cement this position.

This is now a very active research area, not just for mathematicians but also for physicists and philosophers. Thus Helmholtz, primarily a physicist, was already, in 1876, considering the implications of a non-Euclidean geometry [23]. Minkowski is a mathematician but refers to results from experimental physics and so relates geometry to physics. [33]. Russell has developed geometry from a philosophical viewpoint [47]. All the great recent mathematicians, including, Gauss, Riemann, Hilbert and Poincaré, have made major contributions to this theory.

Poincaré takes up the question of whether non-Euclidean geometry is *true* in his 1891 essay *Non-Euclidean Geometries* [41, p.104]:

"What, then, are we to think of the question: Is Euclidean geometry true?

"It has no meaning.

"We might as well ask if the metric system is true and the old weights and measures are false, if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another, only more convenient.

"Now Euclidean geometry is, and will remain, the most convenient"

Non-Euclidean geometry is studied for its own sake, as a beautiful theory. There is also a great deal of speculation on what it tells us about the actual nature of space. Is space flat or curved?

Chapter 6

Space and Geometry

6.1 Connection of Geometry to the Real World

The earliest form of geometry, even before the Greeks, arose out of a need to measure and build. Thus geometry started out as a real world tool. The Greeks then developed it into an academic discipline in its own right. As geometry has developed there have always been connections to reality. [17] However much of the work up to the middle of the nineteenth century has been on developing geometry as an abstract discipline.

As theories of different geometries such as hyperbolic and spherical geometries became accepted as being internally self consistent the question was asked which of these geometries matches the actual behaviour of space.

Thus we change from asking whether a geometry is a possible mathematical model to asking whether it is **the true model** of our universe.

There are three possibilities which are characterised by the curvature of space:

- Zero curvature, flat space, sometimes called parabolic. This is the geometry in which Euclid's fifth postulate holds
- Negative curvature, hyperbolic space. Lobachevskii space where fifth postulate is denied but other 4 postulates retained.

[17] Some obvious examples include: Pythagoras' theorem, the renaissance artists work on perspective, cartography, Monge's descriptive geometry; although many more examples could be cited

- Positive curvature, spherical or Riemannian space. This must be a finite geometry as we must also reject Euclid's 2nd postulate. It is usually described as finite but unbounded.

6.2 Possibilities of Higher Dimensions

Gauss was the first to develop a theory of curvature of space but it was specific to a 2 dimensional surface with the radius of curvature in an orthogonal dimension. This is most easily visualised for spherical geometry where we can consider a two dimensional geometry on the surface of a sphere.

Helmholtz describes such geometry in his 1876 article *On the Origin and Meaning of Geometrical Axioms* [23, p.54].

"Let us, as we logically may, suppose reasoning beings of only two dimensions live and move on the surface of some solid body. We will assume that they have not the power of perceiving anything outside this surface, but that upon it they have perceptions similar to ours. If such beings worked out a geometry, they would of course assign only two dimensions to their space. They would ascertain that a point in moving describes a line, and that a line in moving describes a surface. But they could as little represent to themselves what further spatial construction would be generated by a surface moving out of itself, as we can represent what would be generated by a solid moving out of the space we know."

This same theme is then taken up and further developed be E. A. Abbott in his little book *Flatland* [1], originally published in 1884, where he imagines two dimensional characters: Triangles, Squares (himself), Pentagons etc and the difficulties they would face in visualising a third dimension, even when a visitor, a Sphere, arrives from a three dimensional world, *spaceland* , and tries to explain it.

What both these authors are hinting at is that it is perfectly feasible that we live in a three dimensional world wrapped around some four dimensional object and we could be blissfully unaware of this.

Newcombe also expands on this view [35, p.94], when he describes a potential universe positively curved in a fourth dimension in such a way that a geodesic returns to its starting point after a distance $2D$.

6.3 Einstein's Relativity and Minkowski Space-Time

In 1905 Einstein published his Special Theory of Relativity under the unassuming title *On the electrodynamics of moving bodies* [13]. This followed the experiment of Michelson and Morley in 1888 which showed beyond reasonable doubt that the speed of light is constant regardless of the speed of the emitter (Appendix C).

Einstein started from only two assumptions: [13, p.4]

1. "The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of co-ordinates in uniform translatory motion."
2. "Any ray of light moves in the *stationary* system of co-ordinates with the determined velocity c, whether the ray be emitted by a stationary or by a moving body."

From which he was able to show that two observers in relative motion will obtain different results if measuring the length of the same object. Of particular interest to us is his rule for combining velocities. [13, p.12]

$$V = \frac{v + w}{1 + vw/c^2}$$

Or, if we chose units such that the speed of light, c, is one,

$$V = \frac{v + w}{1 + vw}$$

Thus velocities combine in the same way as do distances in hyperbolic space. So relativistic vector space is a form of hyperbolic space but not exactly the same as that developed by Lobachevskii and others since here we are talking about a finite space with $v < c$.

Minkowski further developed Einsteins ideas in his lecture *Space and Time* [33].

He starts "Gentlemen. The views of space and time which I want to present to you arose from the domain of experimental physics, and therein lies their strength." A clear reference to the Michelson Morley experiment. He continues "Their tendency is radical. From now onwards space by itself and time by itself will recede completely to become mere shadows and only a type of union of the two will stand independently on its own"

He then proceeds, through purely geometrical arguments, to show that we can no longer think purely of three spatial dimensions when observers are in relative motion. Instead we must think in four dimensional "space-time". In three dimensional Euclidean space, the measure of length

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

is conserved. This is the geodesic or shortest distance between the two points. Minkowski showed that if a body is in uniform motion relative to an observer the conserved quantity, or geodesic, becomes

$$\sqrt{c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2}$$

so that in one spatial dimension the equation of motion becomes

$$c^2t^2 - x^2 = constant$$

which is a hyperbola. So the geometry of space-time is hyperbolic between any spatial dimension and the time dimension but not necessarily between any two spatial dimensions.

Again this provides an example of a form of hyperbolic space but not the same form as defined by Lobachevskii. Note also that both Einstein's Special Relativity and Minkowski's Space Time are only relevant when two observers are in relative motion. The effects are zero when the observers are at rest relative to each other - the case that Lobachevskii and others were studying.

6.4 Is Space Euclidean?

One of the big questions now becomes *is space flat or curved and how can we tell?*.

Lobachevskii [29] states that, as far as he can see, it will never be possible to measure the angles of a triangle to sufficient accuracy to determine if their sum is truly less than π.

“Hence there is no means, other than astronomical observations, to use for judging the exactitude which pertains to the calculations of the ordinary geometry.

“This exactitude is very far reaching, as I have shown in one of my investigations, so that, for example, in triangles whose sides are attainable for our measurement[18], the sum of the three angles is not indeed different from two right-angles by the hundredth part of a second.” [29, p.45]

We have already mentioned Newcombe in Section 6.2, who develops a theory of a universe positively curved in a fourth dimension in such a way that a geodesic returns to its starting point after a distance $2D$. He concludes [35, p.94]:

“It may also be remarked that there is nothing within our experience which will justify a denial of the possibility that the space in which we find ourselves may be curved in the manner here supposed. It might be claimed that the distance of the farthest visible star is but a small fraction of the greatest distance, D, but nothing more.”

We have already seen that a non-flat geometry implies an absolute standard of

[18] This is a triangle with as base the diameter of the earth's orbit around the sun (approx. 300 million km) and a distant star as apex

length but, since we are unable to detect any difference in the sum of the angles of the largest triangles from 180°, that standard of length must be greater than the distance to the furthest stars.

We are still no closer to knowing whether the overall geometry of the universe is Euclidean or hyperbolic or spherical and will probably never know. In fact Poincaré sums up the *impossibility* of ever finding out in *Science and Hypotheses* [43, p.305]

"I have on several occasions in the preceding pages tried to show how the principles of geometry are not experimental facts, and that in particular Euclid's postulate cannot be proved by experiment. ...

"... If Lobachevskii's geometry is true, the parallax of a very distant star will be finite. If Riemann's [spherical] is true, it will be negative. ... But what we call a straight line in astronomy is simply the path of a ray of light. If, therefore we were to discover negative parallaxes, or to prove that all parallaxes are higher than a certain limit, we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the law of optics, and suppose that light is not rigorously propogated in a straight line. It is needless to add that everyone would look upon this solution as the more advantageous. Euclidean geometry, therefore, has nothing to fear from fresh experiments."

What Poincaré does not say is that it is even more impossible (if such a concept is allowed) to determine by measurements that the universe is Euclidean, for to **prove** flatness we must obtain an **exact** result for parallax which is impossible since every measurement will be subject to experimental error.

Therefore we may justly paraphrase Poincaré and say "non-Euclidean geometry, therefore, has nothing to fear from fresh experiments".

Chapter 7

Summary

In this essay we have followed the course of geometry up till 1910. We saw how this essentially started with the ancient Greeks when Euclid produced what many still consider to be the definitive work *The Elements* [15]. There was not much change until the start of the nineteenth century. Then geometry took two distinct paths.

7.1 Path 1, Projective Geometry

This started as a descriptive approach through the work of Poncelet and others and then became algebraic, initially through Möbius who also introduced homogeneous coordinates. It introduced concepts such as duality and conservation of cross ratio (defined for distances and angles). It was originally non-metrical but Cayley was able to define a metric for both distance and angles through a concept called *The Absolute* and in this way Euclidean geometry became a sub-set of projective geometry.

Poncelet is generally credited with having started this work on projective geometry, even though earlier mathematicians had already worked out properties of cross ratio, so the whole development took place between 1822 and 1859.

7.2 Path 2, Non-Euclidean Geometry

The development of non-Euclidean geometry started earlier. One could even say that it started as soon as *The Elements* appeared but the first serious start was the work of Saccheri (1733) and Lambert (1766). But Saccheri's work contained an error and Lambert's was not known until it was republished by Engel and Stäckel in 1895 [14]. So the first real introduction of non-Euclidean geometry that anyone noticed was the paper by Lobachevskii originally published, but ignored, in Russian in 1829 then in German in 1840 [29], closely followed by that of J. Bolyai in 1832. So the start of non-Euclidean geometry is generally accepted to be in 1829 with Lobachevskii's Russian paper.

It was not long before Gauss had started, and Riemann generalised, a theory of curvature which was all in place by 1854. Following this there was significant work by Beltrami, Poincaré and Klein to develop models for visualising this new geometry and devise metrics which would combine non-Euclidean geometry with projective geometry. This metric became known as the Cayley-Klein metric because it built on the earlier work of Cayley and the *absolute*. It was published in 1871 and with that all of geometry became united within projective geometry.

7.3 Subsequent progress

The above, very brief, summary misses out most of the mathematicians who also made significant contributions to geometry. In doing so it perhaps masks the fact that throughout the nineteenth century and still up to 1910 both branches of geometry were the subject of intense research and this is expected to continue. Thus the developments in the nineteenth century transformed geometry from a sleepy science or, at best, a tool to teach a rigorous approach to problem solving into a very active area of teaching and research.

This includes the axiomatic approach, complete duality between geometric objects, application of group theory to geometry, the search for new geometries and the continual question of whether the geometry of the universe is, or is not, Euclidean.

Appendix A

Pole and Polar for a Circle

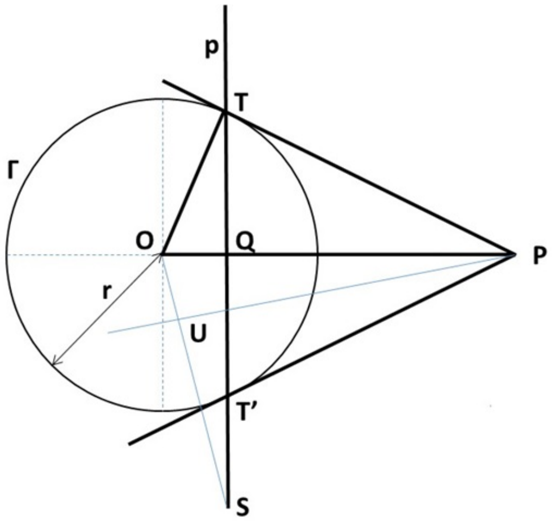

Figure A.1: Pole , P, and Polar, p, for a circle where the pole is outside the circle

We start with the diagramme in Figure A.1 for the special case where the conic is a circle, centre O, and the pole, P, is outside the circle. Draw the tangents from P to touch the circle at t and T'. Draw the line TT' and call this the polar, p.

By symmetry, p is perpendicular to OP. Let them cut at the point Q. Then by similar triangles we have $\frac{OQ}{OT} = \frac{OT}{OP}$ and therefore $OP.OQ = OT^2 = r^2$. The point Q is called the reciprocal point to P.

We can use this as an alternative definition of pole and polar for the circle:-

Definition A.1. Given a circle, Γ of radius r, centre O, and a point P, the pole.

Mark the point Q on OP such that $OP.OQ = r^2$. The polar is the line through Q perpendicular to OP.

Now consider any point, S, on p and draw the line OS. Drop a perpendicular from P onto OS, to meet OS at U. By similar triangles we have $\frac{OU}{OP} = \frac{OQ}{OS}$. Therefore $OU.OS = OP.OQ = r^2$. Therefore the line UP is the polar corresponding to the new pole S from Definition A.1, above. Further we see that the polar corresponding to any point on p will pass through P.

Now consider Figure A.2 when the pole is inside the circle. Again identify Q, the reciprocal point to P, i.e. $OP.OQ = r^2$. The polar is the perpendicular to OP through Q by Definition A.1. We can again show that any new pole, S, on p will define a new polar passing through P.

This shows the relationship between pole and polar for a circle which applies whether the pole is inside or outside the circle. If the pole is outside the circle the polar will pass inside and vice versa.

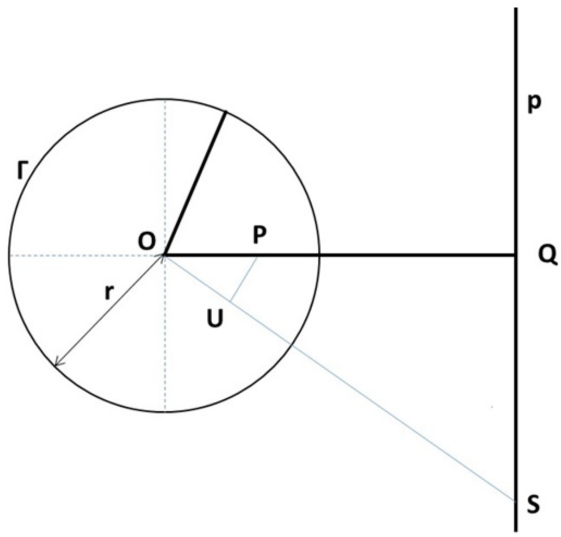

Figure A.2: Pole and Polar for a circle where the pole is inside the circle

There are 3 special cases to be considered.

1. If P is on the circle the polar is the tangent at P.
2. If P is at the centre of the circle the polar is the circle at infinity.
3. If P is infinitely far from the centre, the pole is the diameter of the circle perpendicular to OP.

Appendix B

Models of Hyperbolic Geometry

Over a *small* region, hyperbolic geometry appears identical to Euclidean geometry. For example the sum of the angles of a triangle approaches two right angles as the length of the longest side of the triangle becomes small and approaches zero. However *small* is a comparative term since we are unable to detect any differences in the largest triangles we can measure. Clearly then, if we want to see some of the special effects in non-Euclidean geometry we will need to be able to visualise what happens at extreme distances and for extremely large triangles. This is where the models to be described are valuable since they both map the whole of space into the open unit disc, $x^2 + y^2 < 1$. The boundary at $x^2 + y^2 = 1$ is not a part of either model because this corresponds to the points at infinity.

Both models were originally developed by Beltrami and published in two articles in 1868 [6] [7].

The two dimensional versions of both these models are described below.

B.1 Projective, Beltrami Disc, Klein Disc Model

This starts with a geodetic projection, Figure B.1, from the infinite hyperbolic plane, assumed to be at height a above the unit sphere, onto the half sphere $x^2 + y^2 + z^2 = 1, \quad z > 0$. This is followed by a vertical projection from the half sphere onto the plane $z = 0$. Clearly the whole of the infinite plane will be mapped into the open unit disc.

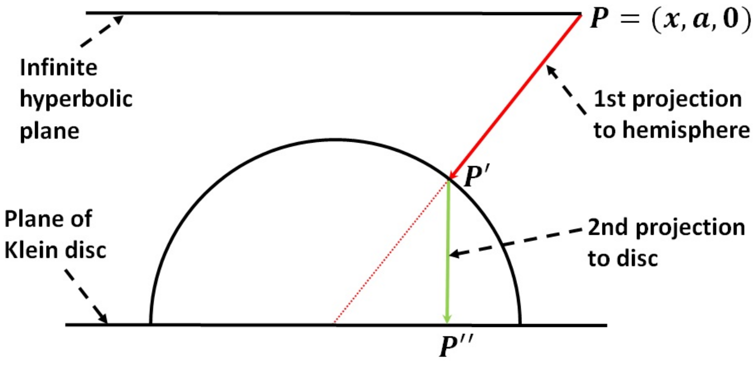

Figure B.1: Projections to obtain Beltrami (Klein) disc. First a geodetic projection from the infinite plane to the surface of a hemisphere, then a vertical projection to the disc.

Some properties of the Klein disc model are:

- Geodesics in the hyperbolic plane are represented by straight lines, chords, in the model.
- Angles are not preserved and circles are not mapped into circles.
- Parallel lines (in the sense of Lobachevskii) meet on the perimeter of the disc (at infinity).

This is shown in Figure B.2 where lines BA, CA and DA are parallel. All lines in this diagramme have been drawn as chords, extended to touch the periphery circle, so that they represent lines stretching to infinity in both directions. Triangle JKL is formed from 3 geodesics: AC, GH and EF. Naturally the sum of the angles is $180°$ in the model whereas the angles in the hyperbolic plane will sum to less than two right angles.

The metric of length, Cayley-Klein metric, can be considered with respect to geodesic GH. The perimeter of the disc is the *absolute* so that G and H are both ideal points. Then the length of the segment JK, which we will call $d(JK)$, is

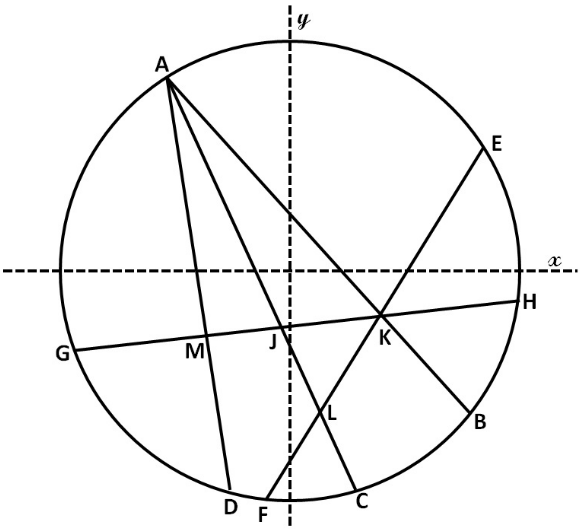

Figure B.2: Beltrami (Klein) disc, all straight lines represent geodesics, BA, CA and DA are parallel (meet at infinity)

given in terms of the cross ratio of the four points G, J, K, H by [20, p.233]

$$d(JK) = -\frac{1}{2}\log\left(\frac{GJ}{GK} \div \frac{HJ}{HK}\right)$$

Note that this implies

$$d(MJ) = -\frac{1}{2}\log\left(\frac{GM}{GJ} \div \frac{HM}{HJ}\right)$$

$$d(MK) = -\frac{1}{2}\log\left(\frac{GM}{GK} \div \frac{HM}{HK}\right)$$

so that

$$d(MK) = d(MJ) + d(JK)$$

and the distance metric is correctly additive.

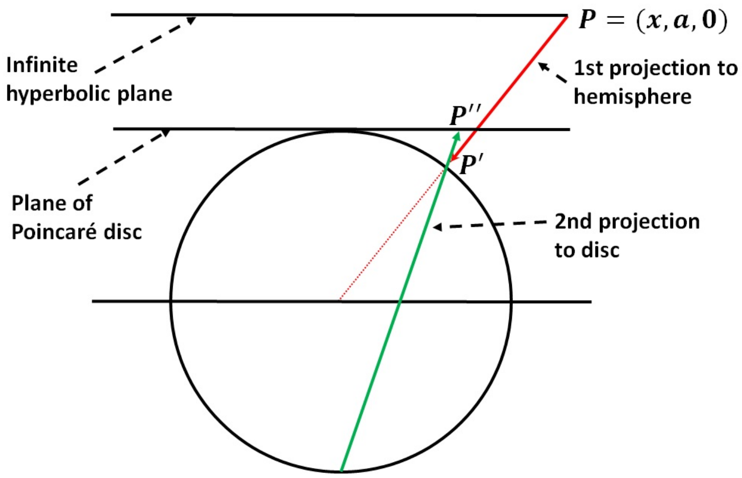

Figure B.3: Projections to obtain Poincaré disc. First a geodetic projection from the infinite plane to the surface of a hemisphere, then a stereographic projection from the base of the sphere to the tangent plane at $z = 1$.

B.2 Conformal, Poincaré Disc Model

The starting point is exactly the same as for the Klein disc but the second projection, Figure B.3, is upwards from the base of the sphere. This then needs scaling to obtain the unit disc. It has the following properties.

- It is conformal meaning that it preserves angles.
- Circles are mapped into circles although their centres are not in general mapped to the centre of the mapped circle.
- Geodesics are mapped into segments of circles meeting the perimeter at right angles.
- A length x is mapped into a length $\tanh^{-1}(x)$

Figure B.4 shows essentially the same configuration as Figure B.2 except that the former shows it in the Klein disc, the latter in the Poincaré disc.

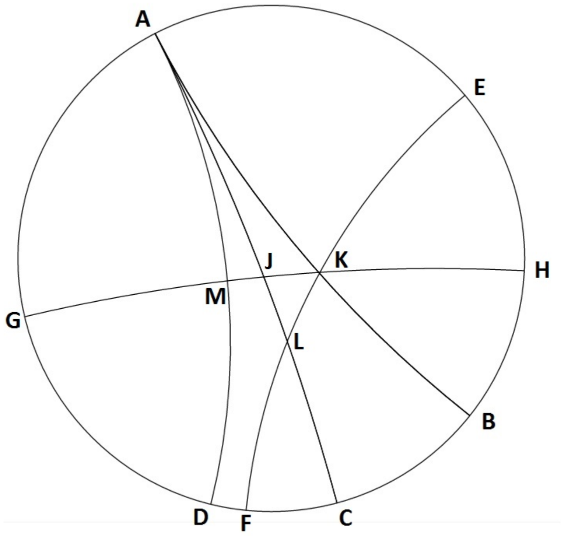

Figure B.4: Poincare disc, all geodesics are arcs of circles meeting the perimeter at right angles, BA, CA and DA are parallel (meet at infinity)

A length is defined in exactly the same way as in the Klein disc, except that the distances to be used in the cross ratio are straight line Euclidean distances, not arcs of the geodesic curves.

Appendix C

Michelson Morley Experiment

In the nineteenth century there was general agreement that light was a wave form rather than a stream of particles. This was based on demonstrations of interference and diffraction of light that are easy to explain for a wave form but not for particles. The problem then was that, if light is a wave, there must be a medium in which it is propagated. This medium was called the luminiferous aether and needed to posses strange properties: it had to be highly rigid to account for the high speed of light and it had to have negligible resistance to the motion of the planets. There were other difficulties with the aether theory including Maxwell's equations [31, p.579] which pointed to a constant speed of light whereas theory said that the speed of light should be constant relative to the aether so the local speed should be a function of the earth's speed through the aether.

Michelson and Morley [32] set out to measure the speed of the earth through the aether by measuring the effect this had on the speed of light.

The experiment was very simple in principle: two beams of light are projected at right angles through the same distance, reflected back and the time difference between the two beams is measured. Assuming that the motion through the aether is not in a direction bisecting the two beams there should be a time difference which can be used to measure the speed of the earth through the aether.

The experimental reality involved splitting a single beam into two beams, reflecting each back and forth many times before recombining them and observing the interference fringes. The whole apparatus, which must have weighed several tons, was floating on a bowl of mercury so that it could be rotated through different angles to observe any changes due to the different directions of the two beams relative to the aether.

They measured changes in the difference between the two transit times rather than measuring the differences directly. This meant that they could allow the two returning beams to interfere and count the interference fringes. These fringes were counted as the instrument was rotated giving a very precise measure of any change in the speed of light due to the different orientation of the two beams.

The main source of the earth's movement through the aether is the rotation of the earth around the sun which gives a speed of approximately 3×10^4 m/s or 0.01 percent of the speed of light. There should also be a smaller effect due to the spin of the earth which is approximately 460 m/s at the equator.

The final apparatus was certainly sensitive enough to detect any change due to the earth's rotation around the sun but detected nothing. The experiments were carried out at different times of the day to identify any small differences due to the earth's rotation. Again nothing was found. The authors conceded that there is a small possibility that the motion of the solar system through the universe could be canceling out the speed of the earths trajectory so proposed repeating the experiment in three monthly intervals. Naturally nothing was found.

Michelson and Morley finally concluded that they could not measure any motion of the earth through the aether. They rejected the argument that the earth locally drags the aether along with it as being inconsistent with other theories and results.

This has been explained at some length because it is one of the most famous "null result" experiments and because this led directly to the realisation that the speed of light was indeed constant and independent of the speed of the emitter and the receiver, and hence to Einstein's theory of relativity. In a sense this mirrors the many attempts to prove the parallel postulate by finding a contradiction. These again led to a "null result" which then directly led to the development of a beautiful new geometry.

The immediate consequence of this experiment was not a rejection of the aether but rather an attempt to explain the effect in a way that involved the aether. Lorentz [30] proposed that the motion somehow compresses an object so that its length in the direction of movement through the aether is reduced, but was unable to find any possible mechanism. This is the famous Lorenz contraction

$$l' = l\sqrt{(1 - v^2/c^2)}$$

which was developed by Lorentz on the basis that it fitted the facts and was then re-derived and explained by Einstein as part of the special theory of relativity.

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