To construct a regular pentagon, using the method due to Gauss

This is explained for the 17 sided polygon in Hardy and Wright, section 5.8, pp 57-62, the pentagon is much simpler.

The pentagon is built up from 5 equal chords of a circle with the angle subtended by any chord at the centre equal to 72^o or $2\pi/5$, call this angle θ. See Figure 1

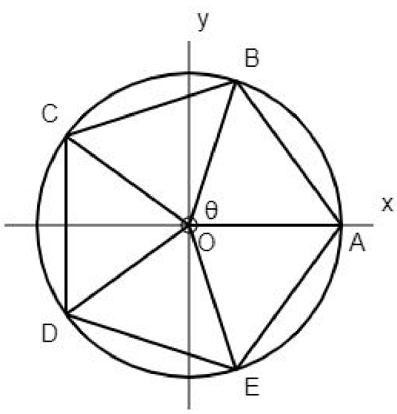

Figure 1: Regular Pentagon

So we need to be able to create an angle of 72^o or $2\pi/5$, using just compass and straight edge. This can be done if we can express the trigonometric ratios of this angle in terms of square roots of integers which we can then easily construct using right angled triangles.

Two approaches are shown, the first is somewhat woolly but avoids us of complex numbers, the second follows Hardy and Wright.

0.1 Woolly approach avoiding complex numbrs

Consider the circle in Figure 1 as a rigid disc with unit weights at the points A, B, C, D, E. Because the weights are uniformly distributed this disc will clearly balance horizontally on a pin at O. Therefore it will also balance on any horizontal edge through O. One such edge is the y axis, normal to OA. Therefore the vertical distance of the 5 weights from the y axis must sum to zero, or:

$$1 + \cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0$$

But, as can be seen in the Figure 1, $\cos\theta = \cos 4\theta$ and $\cos 2\theta = \cos 3\theta$. So that

$$2(\cos\theta + \cos 2\theta) = -1$$

Now go to section entitled “Common continuation”

0.2 More rigorous approach

Consider Figure 1 as a diagram of the regular pentagon in the complex plane, y is the imaginary axis. The vertices are at the points $e^{ik\theta}$, $k = 0, 1, 2, 3, 4$, i.e. at the points z^k where $z = e^{i\theta}$ and $z^5 = 1$.

We need to solve $z^5 = 1$, but since $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, we ignore the trivial solution and solve

$$z^4 + z^3 + z^2 + z = -1$$

Write $e_k = e^{ik\theta}$, i.e. $e_1 = z, e_2 = z^2 \ldots$, and solve

$$e_4 + e_3 + e_2 + e_1 = -1 \tag{1}$$

But

$$e_k = e^{ik\theta} = \cos(k\theta) + i\sin(k\theta) = \cos(2\pi k/5) + i\sin(2\pi k/5)$$

so that

$$e_k + e_{5-k} = 2\cos k\theta$$

and therefore

$$e_1 + e_4 + e_2 + e_3 = 2(\cos\theta + \cos 2\theta) = -1$$

0.3 Common continuation

We have $2\cos\theta + 2\cos 2\theta = -1$. We write $z_1 = 2\cos\theta$, $z_2 = 2\cos 2\theta$, so that $z_1 + z_2 = -1$. We now want to find $z_1 \times z_2$

We use

$$\cos(a \pm b) = \cos a\cos b \mp \sin a \sin b$$

so that

$$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$$

Then

$$z_1 z_2 = 4\cos\theta\cos 2\theta = 2(\cos 3\theta + \cos\theta)$$

But $\cos 3\theta = \cos(2\pi - 3\theta) = \cos(5\theta - 3\theta) = \cos 2\theta$ so that

$$z_1 z_2 = 2(\cos\theta + \cos 2\theta) = -1$$

To summarise, we have $z_1 + z_2 = -1, \quad z_1 \times z_2 = -1$, so z_1 and z_2 are roots of a quadratic equation with sum of the roots $= -1$ and product of the roots $= -1$. This equation must be

$$z^2 + z - 1 = 0$$

The positive root is $z = (\sqrt{5} - 1)/2 = 2\cos\theta$ so that

$$\cos\theta = \frac{\sqrt{5}-1}{4}$$

A quick check on calculator gives $\theta = 72^o$ (OK).

Further the second root is $\frac{-\sqrt{5}-1}{2} = 2\cos 2\theta$, giving $2\theta = 144^o$.

1 To draw this regular pentagon with ruler and compass

See Figure 2

1. Draw OA of length 2, continue to A' so that $OA' = 4$
2. Draw perpendicular at A, length 1 to B
3. Connect OB of length $\sqrt{5}$ and step off 1 from B to C, so that $OC = \sqrt{5} - 1$
4. Draw the circle, centre O, passing through C (radius $\sqrt{5} - 1$)
5. Now draw CD, perpendicular to OB
6. From O draw an arc of radius 4 to meet CD at E. Use OA' as reference.
7. Triangle OCE is right angled with $OC = \sqrt{5} - 1$ and $OE = 4$, so that
$$\cos(COE) = \frac{\sqrt{5}-1}{4}$$
8. OE cuts the circle at F so that CF is one side of the regular pentagon

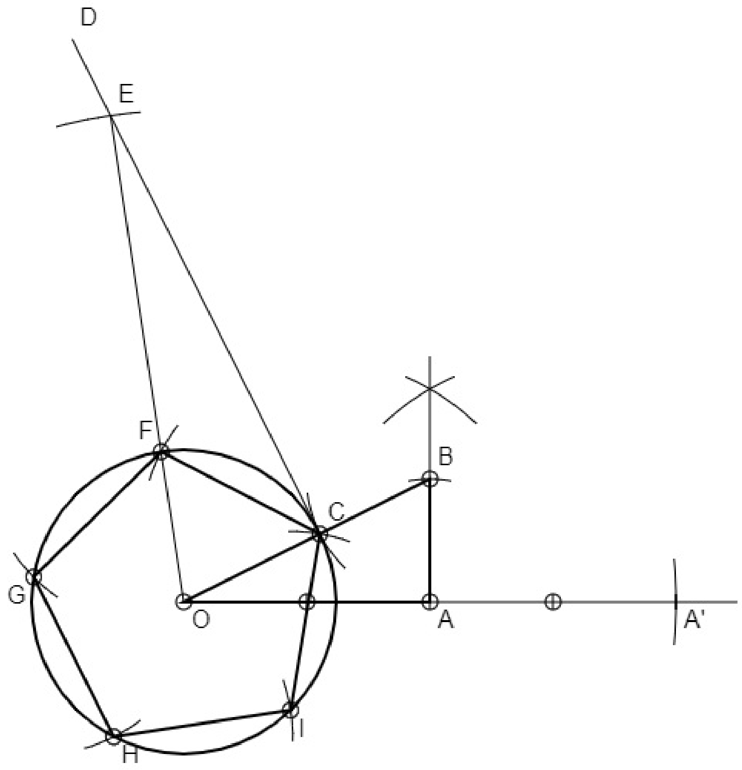

Figure 2: Ruler and compass construction